Tutorial 5

Feb.25,2016

1. **Well-posedness of the following diffusion equations:**

$$\begin{cases} \partial_t u - k\partial_x^2 u = f(x,t), -\infty < x < \infty, t > 0 \\ u(x, t=0) = \phi(x), -\infty < x < \infty \end{cases}$$

The well-posedness has three ingredients:

(a) Existence: there exists at least one solution. We have shown that

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds$$

is a solution, where $S(z,\tau)$ is the heat kernel.

(b) Uniqueness: there exists at most one solution. And we have shown that the above solution is unique.

(c) Stability: if the data changes a little, then u also changes only a little.And we will show that the above problem is stable in the sense of uniform norm.

Suppose that $u_i(x,t)$ is the solution with the source $f_i(x,t)$ and the initial data $\phi_i(x)$, $i = 1, 2$. Set $u(x,t) = u_1(x,t) - u_2(x,t)$, $f(x,t) = f_1(x,t) - f_2(x,t)$ and $\phi(x) = \phi_1(x) - \phi_2(x)$, thus $u(x,t)$ is a solution of

$$\begin{cases} \partial_t u - k\partial_x^2 u = f(x,t), -\infty < x < \infty, t > 0 \\ u(x, t=0) = \phi(x), -\infty < x < \infty \end{cases}$$

thus

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds$$

by the solution formula.

For any $-\infty < x < \infty, 0 \le t \le T$, then

$$\begin{aligned} |u(x,t)| &\le \int_{-\infty}^{\infty} S(x-y,t)|\phi(y)|dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)|f(y,s)|dyds \\ &\le \int_{-\infty}^{\infty} S(x-y,t) \max_{-\infty<y<\infty} |\phi(y)|dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) \max_{-\infty<y<\infty, 0\le t\le T} |f(y,s)|dyds \\ &\le \|\phi\| \int_{-\infty}^{\infty} S(x-y,t)dy + ||f||_T \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)dyds \\ &\le \|\phi\| + \|f\|_T \int_0^t ds \qquad \text{by} \int_{-\infty}^{\infty} S(x,t)dx = 1 \\ &\le \|\phi\| + T\|f\|_T \end{aligned}$$

Hence

$$\|u\|_T \le \|\phi\| + T\|f\|_T$$

So if $\|f\|_T$ and $\|\phi\|$ are small, then $\|u\|_T$ is small.

2. The compatibility of the following IBVP:

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = 0, \quad x > 0, t > 0 \\ u(x, t = 0) = \phi(x), \quad x > 0 \\ \partial_t u(x, t = 0) = \psi(x), \quad x > 0 \\ u(x = 0, t) = h(t), \quad t > 0 \end{cases}$$

If we want that the solution of the problem is continously diffentiable at the corner $(x = 0, t = 0)$,thus the given conditions must coincide with each other:

$$\phi(0) = h(0)$$

$$\psi(0) = h'(0)$$

If the compatibility conditions are not satisfied, then there will be a singularity on the characteristic line emanating from the corner.

3. **Method of characteristic coordinates on p71**
Theorem 1(on P69): The unique solution of

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = f(x, t), -\infty < x < \infty, t > 0 \\ u(x, t = 0) = \phi(x), -\infty < x < \infty \\ \partial_t u(x, t = 0) = \psi(x), -\infty < x < \infty \end{cases}$$

is

$$u(x, t) = \frac{1}{2}[\phi(x + ct) - \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy + \frac{1}{2c} \iint_{\Delta} f(y, s) dy ds$$

where Δ is the characteristic triangle.

Proof: Use the characteristic coordinates:

$$\xi = x + ct$$

$$\eta = x - ct$$

Then we have

$$Lu = \partial_t^2 u - c^2 \partial_x^2 u = -4c^2 \partial_{\xi\eta}^2 u = f(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c})$$

We integrate this equation with respect to η, leaving ξ as a constant.Thus $\partial_\xi u = -\frac{1}{4c^2} \int^\eta f d\eta$. Then we integrate with respect to ξ to get

$$u = -\frac{1}{4c^2} \int^\xi \int^\eta f d\eta d\xi$$

The lower limits of integration here are arbitrary: They correspond to constants of integration.Now we make a particular choice of the lower limits and find a particular solution:

$$\begin{aligned} u(P_0) = u(x_0, y_0) &= -\frac{1}{4c^2} \int_{\eta_0}^{\xi_0} \int_{\xi}^{\eta_0} f(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}) d\eta d\xi \\ &= \frac{1}{4c^2} \int_{\eta_0}^{\xi_0} \int_{\eta_0}^{\xi} f(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}) d\eta d\xi \\ &= \frac{1}{4c^2} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(x, t) J dx dt \quad \text{(change of variables)} \\ &= \frac{1}{4c^2} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(x, t) 2c dx dt \\ &= \frac{1}{2c} \iint_{\Delta} f(x, t) dx dt \end{aligned}$$

This is precisely Thereom 1. (See the figures on the books and you will understand clearly.)